1. Pentru sistemul sferic de coordonate, componentele contravariante ale vectorilor sunt $(1,0,0)$, $(0,1/R,0)$, $(0,0,1/R\cdot\sin\theta)$.
Demonstrați că acești trei vectori sunt ortogonali.

Rezolvare:

Pentru sistemele ortogonale baza reciprocă este aceeași ca și baza principală, iar $g_{ik} = \delta_{ik}$.

$$\vec{A} = A^{(1)}\vec{e}_1 + A^{(2)}\vec{e}_2 + A^{(3)}\vec{e}_3;\ \vec{B} = B^{(1)}\vec{e}_1 + B^{(2)}\vec{e}_2 + B^{(3)}\vec{e}_1;$$

$$\vec{c} = c^{(1)}\vec{e}_1 + c^{(2)}\vec{e}_2 + c^{(3)}\vec{e}_3;$$

$$\left(\vec{e}_1, \vec{e}_2, \vec{e}_3\right) \rightarrow \text{principală},$$

$$\left.\begin{array}{l} x_1 = R\sin\theta\cos\varphi,\ x_1 = R\sin\theta\sin\varphi,\ x_3 = R\cos\theta \\ R = q^1 = \sqrt{x_1^2 + x_2^2 + x_3^2},\ \theta = q^2 = \text{arctg}\dfrac{\sqrt{x_1^2 + x_2^2}}{x_3},\ \varphi = q^3 = \text{arctg}\dfrac{x_2}{x_1} \end{array}\right\};$$

$$\left.\begin{array}{l} \tilde{e}_1 = \dfrac{\partial\vec{r}}{\partial q^1} = \dfrac{\partial\vec{r}}{\partial R} = \sin\theta\cos\varphi\vec{i}_1 + \sin\theta\sin\varphi\vec{i}_2 + \cos\theta\vec{i}_3 \\ \tilde{e}_2 = \dfrac{\partial\vec{r}}{\partial q^2} = \dfrac{\partial\vec{r}}{\partial\theta} = R\cos\theta\cos\varphi\vec{i}_1 + R\cos\theta\sin\varphi\vec{i}_2 - R\sin\theta\vec{i}_3 \\ \tilde{e}_3 = \dfrac{\partial\vec{r}}{\partial q^3} = \dfrac{\partial\vec{r}}{\partial\varphi} = -R\sin\theta\cos\varphi\vec{i}_1 + R\sin\theta\sin\varphi\vec{i}_2 \end{array}\right\}.$$

Vedem, deci, că sistemul sferic nu-i cartezian.
Deci vom avea baza:

$$\vec{A} = \sin\theta\cos\varphi\vec{i}_1 + \sin\theta\sin\varphi\vec{i}_2 + \cos\theta\vec{i}_3 = \vec{e}_1;$$

$$\vec{B} = \left(R\cos\theta\cos\varphi\vec{i}_1 + R\cos\theta\sin\varphi\vec{i}_2 - R\sin\theta\vec{i}_3\right)\frac{1}{R} = \vec{e}_2;$$

$$\vec{C} = -R\sin\theta\cos\varphi\vec{i}_1 + R\sin\theta\sin\varphi\vec{i}_2 = \vec{e}_3;$$

$$g_{ik} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \delta_{ik} \quad \text{c.t.d.}$$

2. Pentru funcția $U = U(x,y,z)$ să se exprime grad$U$
a) în coordonate cilindrice;
b) în coordonate sferice.

Rezolvare:

a)

$$gradU = \frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k};$$

$$\left.\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{array}\right\} \quad \left.\begin{array}{l} r = \sqrt{x^2 + y^2} \\ \varphi = \arcsin\dfrac{x}{r} \\ z = z \end{array}\right\};$$

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial\varphi}\frac{\partial\varphi}{\partial x} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial U}{\partial r}\cos\varphi + \frac{\partial U}{\partial\varphi}\frac{\partial\varphi}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{r} = \cos\varphi;$$

$$\frac{\partial\varphi}{\partial x} = \frac{r - x\cos\varphi}{r^2}\frac{1}{\sqrt{1 - \dfrac{x^2}{r^2}}} = -\frac{\sin\varphi}{r};$$

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial r}\cos\varphi + \frac{\partial U}{\partial\varphi}\left(-\frac{\sin\varphi}{r}\right);$$

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial r}\sin\varphi + \frac{\partial U}{\partial\varphi}\frac{\cos\varphi}{r};$$

$$\frac{\partial U}{\partial z} = \frac{\partial U}{\partial z};$$

$$\operatorname{grad} U = \left(\frac{\partial U}{\partial r}\cos\varphi - \frac{\sin\varphi}{r}\frac{\partial U}{\partial \varphi}\right)\vec{i} + \left(\frac{\partial U}{\partial r}\sin\varphi + \frac{\partial U}{\partial \varphi}\frac{\cos\varphi}{r}\right)\vec{j} +$$

$$+\frac{\partial U}{\partial z}\vec{k} = \frac{\partial U}{\partial r}\vec{e}_r + \frac{\partial U}{\partial \varphi}\vec{e}_\varphi + \frac{\partial U}{\partial z}\vec{e}_z;$$

$$\vec{e}_r = \vec{i}\cos\varphi + \vec{j}\sin\varphi; \ \vec{e}_\varphi = -\vec{i}\sin\varphi + \vec{j}\cos\varphi, \ \vec{e}_z = \vec{e}_z.$$

b) De sine stătător.

XII. FORMULA GREEN

1. Cu ajutorul formulei Green să se transforme următoarea integrală.

$$I = \oint_C \sqrt{x^2 + y^2}\,dx + y\left[xy + \ln\left(x + \sqrt{x^2 + y^2}\right)\right]dy.$$

Rezolvare;
Folosind formula Green:

$$\oint_C P(x,y)dx + Q(x,y)dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy;$$

$$\sqrt{x^2 + y^2} = P(x,y);$$

$$y\left[xy + \ln\left(x + \sqrt{x^2 + y^2}\right)\right] = Q(x,y);$$

$$\frac{\partial P(x,y)}{\partial y} = \frac{2y}{2\sqrt{x^2 + y^2}};$$

$$\frac{\partial Q(x,y)}{\partial x} = y\left[y + \frac{1}{x + \sqrt{x^2 + y^2}}\left(1 + \frac{1}{2\sqrt{x^2 + y^2}}2x\right)\right];$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} - y\left[y + \frac{1}{x + \sqrt{x^2 + y^2}}\left(1 + \frac{1}{2\sqrt{x^2 + y^2}}2x\right)\right] = y^2$$

Răspuns:

$$I = \iint_S y^2\,dxdy.$$

2. Folosind formula Green, calculați următoarea integrală curbilinie:

$$\oint_C xy^2 dy - x^2 y dx,$$