III. TENSOR METRIC. NOȚIUNE DE TENSOR. TRANSFORMAREA COMPONENTELOR TENSORIALE. TENSORI DE RANGURILE I ȘI II

1. Demonstrați că energia cinetică a unui corp solid, ce are un punct fix și se mișcă cu viteza unghiulară momentană ω, este

$$T = \frac{1}{2}\omega^2 I_{\omega\omega},$$

$I_{\omega\omega}$ - momentul de inerție al corpului față de axa de rotație.

Rezolvare:

Energia cinetică:

$T = \frac{1}{2}\sum_{n=1}^{N} m_n \vec{V}_n^2$, folosim formulele Euler

$$\vec{V}_n = \vec{\omega} \times \vec{r}_n;$$

$$T = \frac{1}{2}\sum_{n=1}^{N} m_n (\vec{\omega} \times \vec{r}_n)^2 = \frac{1}{2}\sum_{n=1}^{N} m_n (\vec{\omega} \times \vec{r}_n)(\vec{\omega} \times \vec{r}_n). \quad (1)$$

Folosim formulele:

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{A} \cdot \vec{C}) - \vec{C} \cdot (\vec{A} \cdot \vec{B});$$

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{B} \times \vec{C}) \cdot \vec{A} = (\vec{C} \times \vec{A}) \cdot \vec{B}; \quad (2)$$

$$T = \frac{1}{2}\sum_{n=1}^{N} m_n \omega[\vec{r}_n \times (\vec{\omega} \times \vec{r}_n)] = \frac{1}{2}\sum_{n=1}^{N} m_n \omega[\omega(\vec{r}_n)^2 - \vec{r}_n(\vec{r}_n \omega)] =$$

$$= \frac{1}{2}\sum_{n=1}^{N} m_n [\omega^2 (\vec{r}_n)^2 - (\omega \cdot \vec{r}_n)^2]. \quad (3)$$

Notăm $\vec{\omega}^0 = \frac{\vec{\omega}}{\omega}$ și considerând expresia pentru momentul de inerție (pr.2)

$$I_{nn} = \sum_{n=1}^{N} m_n \left[x_l^{(n)} x_l^{(n)} - \left(x_i^{(n)} u_i \right)^2 \right];$$

$$T = \frac{1}{2}\omega^2 \sum_{n=1}^{N} m_n \left[\vec{r}_n^{\,2} - \left(\omega^0 \cdot \vec{r}_n^{\,2} \right) \right] = \frac{1}{2}\omega^2 I_{\omega\omega},$$

$I_{\omega\omega}$ - tensorul momentului de inerție.

2. Aflați expresia pentru momentul de inerție al sistemului punctelor materiale față de axa (u) cu vectorul unitar $\vec{u}$.

Rezolvare (de sine stătător).

3. În sistemul cartezian de coordonate $K(\vec{i}_1, \vec{i}_2, \vec{i}_3)$ sunt date componentele tensorului de rangul doi:

$$\|A^{ik}\| = \|A_i^k\| = \|A_k^i\| = \|A_{ik}\| = \begin{Vmatrix} 1 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 2 \end{Vmatrix}.$$

Vectori de bază ai sistemului de coordonate K' se exprimă ca:

$$\vec{e}_1 = -\vec{i}_1 + \vec{i}_3; \quad \vec{e}_2 = \vec{i}_1 + \vec{i}_2; \quad \vec{e}_3 = \vec{i}_3.$$

Să se afle componentele covariante, contravariante și mixte în sistemul de coordonate K' (A'^{ik}, A'_{ik}, A'^{k}_{i}, A'^{i}_{k} - ?).

Rezolvare:

Trecerea de la un sistem de coordonate la altul $K' \to K$ se efectuează conform legii:

$$\vec{e}'_i = \alpha^k_{i'} \vec{e}_k. \quad (1)$$

Deci $A'_{ik} = \alpha^l_{i'} \alpha^m_{k'} A_{lm}$, unde coeficienții $\alpha^l_{i'}$ sunt coeficienții transformării directe pe care-i putem afla din matricea de transformare a bazelor (1):

$$\alpha^l_{i'} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2)$$

Vom avea:

$$A'_{11} = \alpha^1_{1'}\alpha^1_{1'}A_{11} + \alpha^1_{1'}\alpha^2_{1'}A_{12} + \alpha^1_{1'}\alpha^3_{1'}A_{13} +$$
$$+\alpha^2_{1'}\alpha^1_{1'}A_{21} + \alpha^2_{1'}\alpha^2_{1'}A_{22} + \alpha^2_{1'}\alpha^3_{1'}A_{23} +$$
$$+\alpha^3_{1'}\alpha^1_{1'}A_{31} + \alpha^3_{1'}\alpha^2_{1'}A_{32} + \alpha^3_{1'}\alpha^3_{1'}A_{33} = 1;$$
$$A'_{12} = \alpha^1_{1'}\alpha^1_{2'}A_{11} + \alpha^1_{1'}\alpha^2_{2'}A_{12} + \alpha^1_{1'}\alpha^3_{2'}A_{13} +$$
$$+\alpha^2_{1'}\alpha^1_{2'}A_{21} + \alpha^2_{1'}\alpha^2_{2'}A_{22} + \alpha^2_{1'}\alpha^3_{2'}A_{23} +$$
$$+\alpha^3_{1'}\alpha^1_{2'}A_{31} + \alpha^3_{1'}\alpha^2_{2'}A_{32} + \alpha^3_{1'}\alpha^3_{2'}A_{33} = -1;$$
$$A'_{13} = \alpha^1_{1'}\alpha^1_{3'}A_{11} + \alpha^1_{1'}\alpha^2_{3'}A_{12} + \alpha^1_{1'}\alpha^3_{3'}A_{13} +$$
$$+\alpha^2_{1'}\alpha^1_{3'}A_{21} + \alpha^2_{1'}\alpha^2_{3'}A_{22} + \alpha^2_{1'}\alpha^3_{3'}A_{23} +$$
$$+\alpha^3_{1'}\alpha^1_{3'}A_{31} + \alpha^3_{1'}\alpha^2_{3'}A_{32} + \alpha^3_{1'}\alpha^3_{3'}A_{33} = 2;$$
$$A'_{21} = \alpha^1_{2'}\alpha^1_{1'}A_{11} + \alpha^1_{2'}\alpha^2_{1'}A_{12} + \alpha^1_{2'}\alpha^3_{1'}A_{13} +$$
$$+\alpha^2_{2'}\alpha^1_{1'}A_{21} + \alpha^2_{2'}\alpha^2_{1'}A_{22} + \alpha^2_{2'}\alpha^3_{1'}A_{23} +$$
$$+\alpha^3_{2'}\alpha^1_{1'}A_{31} + \alpha^3_{2'}\alpha^2_{1'}A_{32} + \alpha^3_{2'}\alpha^3_{1'}A_{33} = -2;$$
$$A'_{31} = \alpha^1_{3'}\alpha^1_{1'}A_{11} + \alpha^1_{3'}\alpha^2_{1'}A_{12} + \alpha^1_{3'}\alpha^3_{1'}A_{13} +$$
$$+\alpha^2_{3'}\alpha^1_{1'}A_{21} + \alpha^2_{3'}\alpha^2_{1'}A_{22} + \alpha^2_{3'}\alpha^3_{1'}A_{23} +$$
$$+\alpha^3_{3'}\alpha^1_{1'}A_{31} + \alpha^3_{3'}\alpha^2_{1'}A_{32} + \alpha^3_{3'}\alpha^3_{1'}A_{33} = 1;$$
$$A'_{22} = \alpha^1_{2'}\alpha^1_{2'}A_{11} + \alpha^1_{2'}\alpha^2_{2'}A_{12} + \alpha^1_{2'}\alpha^3_{2'}A_{13} +$$
$$+\alpha^2_{2'}\alpha^1_{2'}A_{21} + \alpha^2_{2'}\alpha^2_{2'}A_{22} + \alpha^2_{2'}\alpha^3_{2'}A_{23} +$$
$$+\alpha^3_{2'}\alpha^1_{2'}A_{31} + \alpha^3_{2'}\alpha^2_{2'}A_{32} + \alpha^3_{2'}\alpha^3_{2'}A_{33} = 7;$$

$$A'_{23} = \alpha^1_{2'}\alpha^1_{3'}A_{11} + \alpha^1_{2'}\alpha^2_{3'}A_{12} + \alpha^1_{2'}\alpha^3_{3'}A_{13} +$$
$$+\alpha^2_{2'}\alpha^1_{3'}A_{21} + \alpha^2_{2'}\alpha^2_{3'}A_{22} + \alpha^2_{2'}\alpha^3_{3'}A_{23} +$$
$$+\alpha^3_{2'}\alpha^1_{3'}A_{31} + \alpha^3_{2'}\alpha^2_{3'}A_{32} + \alpha^3_{2'}\alpha^3_{3'}A_{33} = 4;$$
$$A'_{32} = \alpha^1_{3'}\alpha^1_{2'}A_{11} + \alpha^1_{3'}\alpha^2_{2'}A_{12} + \alpha^1_{3'}\alpha^3_{2'}A_{13} +$$
$$+\alpha^2_{3'}\alpha^1_{2'}A_{21} + \alpha^2_{3'}\alpha^2_{2'}A_{22} + \alpha^2_{3'}\alpha^3_{2'}A_{23} +$$
$$+\alpha^3_{3'}\alpha^1_{2'}A_{31} + \alpha^3_{3'}\alpha^2_{2'}A_{32} + \alpha^3_{3'}\alpha^3_{2'}A_{33} = 3;$$
$$A'_{33} = \alpha^1_{3'}\alpha^1_{3'}A_{11} + \alpha^1_{3'}\alpha^2_{3'}A_{12} + \alpha^1_{3'}\alpha^3_{3'}A_{13} +$$
$$+\alpha^2_{3'}\alpha^1_{3'}A_{21} + \alpha^2_{3'}\alpha^2_{3'}A_{22} + \alpha^2_{3'}\alpha^3_{3'}A_{23} +$$
$$+\alpha^3_{3'}\alpha^1_{3'}A_{31} + \alpha^3_{3'}\alpha^2_{3'}A_{32} + \alpha^3_{3'}\alpha^3_{3'}A_{33} = 2.$$

Deci

$$A'_{ik} = \begin{Vmatrix} 1 & -1 & 2 \\ -2 & 7 & 4 \\ 1 & 3 & 2 \end{Vmatrix}$$

Componentele contravariante:

$$A'^{ik} = g^{il} g^{km} A_{lm} ,$$

deci avem nevoie de tensorul $g^{ik} = \vec e^{\,i} \vec e^{\,k}$, iar pentru aceasta trebuie să calculăm baza reciprocă.

$$\vec e^{\,1} = \frac{\vec e_2 \times \vec e_3}{\vec e_1(\vec e_2 \times \vec e_3)} = -\vec i_1 + \vec i_2;$$

$$\vec e^{\,2} = \frac{\vec e_3 \times \vec e_1}{V} = -\vec i_1;$$

$$\vec e^{\,3} = \frac{\vec e_1 \times \vec e_{31}}{V} = \vec i_3;$$

$$g^{ik} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix};$$

$$A'^{11} = g^{11}g^{11}A_{11} + g^{11}g^{12}A_{12} + g^{11}g^{13}A_{13} + \\ + g^{12}g^{11}A_{21} + g^{12}g^{12}A_{22} + g^{12}g^{13}A_{23} + \\ + g^{13}g^{11}A_{31} + g^{13}g^{12}A_{32} + g^{13}g^{13}A_{33} = 11;$$

$$A'^{12} = g^{11}g^{21}A_{11} + g^{11}g^{22}A_{12} + g^{11}g^{23}A_{13} + \\ + g^{12}g^{21}A_{21} + g^{12}g^{22}A_{22} + g^{12}g^{23}A_{23} + \\ + g^{13}g^{21}A_{31} + g^{13}g^{22}A_{32} + g^{13}g^{23}A_{33} = 10;$$

etc...

$$A'^{ik} = \begin{Vmatrix} 11 & 10 & 20 \\ 17 & 7 & 16 \\ 16 & 12 & 32 \end{Vmatrix}.$$

Penru componentele mixte

$$A'^{i}_{k} = g^{il}A_{lk};$$

$$A'^{1}_{1} = g^{11}A_{11} + g^{12}A_{21} + g^{13}A_{31} = 5;$$

$$A'^{1}_{2} = g^{11}A_{12} + g^{12}A_{22} + g^{13}A_{32} = 7;$$

$$A'^{1}_{3} = g^{11}A_{13} + g^{12}A_{23} + g^{13}A_{33} = 5;$$

$$A'^{2}_{1} = 4;\ A'^{2}_{2} = 4;\ A'^{2}_{3} = 4;\ A'^{3}_{1} = 4;\ A'^{3}_{2} = 8;\ A'^{3}_{3} = 8;$$

$$A'^{i}_{k} = \begin{Vmatrix} 5 & 7 & 5 \\ 4 & 4 & 4 \\ 4 & 8 & 8 \end{Vmatrix}.$$

La fel componentele $A'_{i}{}^{k}$.

4. Sunt daţi vectorii de bază ai sistemului de coordonate $k'(\vec{e}_1, \vec{e}_2, \vec{e}_3)$.

$$\left.\begin{aligned} \vec{e}_1 &= \vec{i}_1 + \vec{i}_2 \\ \vec{e}_2 &= 2\vec{i}_2 \\ \vec{e}_3 &= -\vec{i}_3 \end{aligned}\right\}.$$

Componentele tensorului in sistemul $k(\vec{i}_1, \vec{i}_2, \vec{i}_3)$ sunt

$$\|A^{ik}\| = \begin{Vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{Vmatrix}.$$

Calculaţi componentele covariante şi **contravariante în sistemul** $k'(\vec{e}_1, \vec{e}_2, \vec{e}_3)$.

Rezolvare (de sine stătător).