$$A_2 = 1;\ A_3 = \frac{1+\sqrt{10}}{3};$$

$$\vec{A}^{(3)} = \vec{i}_2 + \left(\frac{1+\sqrt{10}}{3}\right)\vec{i}_3.$$

Celelalte cazuri - de sine stătător.

V. CÂMPURI SCALARE. GRADIENTUL CÂMPULUI

1. Aflați mărimea și direcția gradientului câmpului $U = x^2 + 2y^2 + 3z^2 + xy + 3x - 2y - 6z$ în punctele: a) O(0,0,0); b) A(1,1,1); c) B(2,0,1). În care puncte gradientul câmpului va fi nul?

Rezolvare:

După definiție: $\mathrm{grad}U = \frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k};$

$$\frac{\partial U}{\partial x} = 2x + y + 3;\ \frac{\partial U}{\partial y} = 4y + x - 2;\ \frac{\partial U}{\partial z} = 6z - 6,$$

pentru $x = y = z = 0$;

$$\mathrm{grad}U = 3\vec{i} - 2\vec{j} - 6\vec{k};$$

$$|\mathrm{grad}U| = \sqrt{9 + 4 + 36} = 7;$$

$$\mathrm{grad}U = 0 \text{ pentru: } \begin{cases} 2x + y + 3 = 0 & x = -2 \\ 4y + x - 2 = 0 & y = 1 \\ 6z - 6 = 0 & z = 1 \end{cases}.$$

2. Fie $U = xy - z^2$. Calculați mărimea și direcția **grad**U în punctul M(-9,12,10). Cu ce va fi egală derivata $\frac{\partial U}{\partial l}$ în direcția bisectoarei unghiului xOz?

Rezolvare:

Analog ca în prima problemă:

$$\mathrm{grad}U\big|_M = 12\vec{i} - 9\vec{j} - 20\vec{k};$$

$$|\mathrm{grad}U|\big|_M = 25.$$

La fel cosinusurile directoare față de axele de coordonate:

$$\cos\alpha = \frac{12}{25}; \quad \cos\beta = -\frac{9}{25}; \quad \cos\gamma = -\frac{4}{5};$$

$$\frac{\partial U}{\partial l} = \mathrm{grad}\,U \cdot \vec{l} = \frac{\partial U}{\partial x}\cos\alpha + \frac{\partial U}{\partial y}\cos\beta + \frac{\partial U}{\partial z}\cos\gamma = \frac{3\sqrt{2}}{2};$$

$$\vec{l} = \frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}.$$

3. În ce puncte ale spațiului gradientul câmpului $\mathrm{grad}\,U = x^3 + y^3 + z^3 - 3xyz$ va fi:

a) perpendicular pe axa Oz;
b) paralel cu Oz;
c) egal cu zero?

4. Este dat câmpul scalar: $U = \ln\frac{1}{r}$,

unde $r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$. În ce puncte ale spațiului $Oxyz$ va avea loc egalitatea $|\mathrm{grad}\,U| = 1$?

Rezolvare (de sine stătător).

5. Aflați ungiul φ dintre gradientul câmpului $U = \dfrac{x}{x^2 + y^2 + z^2}$ în punctele A(1,2,2) și B(-3,1,0).

Rezolvare:

$$\mathrm{grad}\,U = \frac{\partial U}{\partial x}\vec{i} + \frac{\partial U}{\partial y}\vec{j} + \frac{\partial U}{\partial z}\vec{k};$$

$$\frac{\partial U}{\partial x} = \frac{y^2 + z^2 - x^2}{\left(x^2 + y^2 + z^2\right)^2}, \quad \frac{\partial U}{\partial y} = \frac{2xy}{\left(x^2 + y^2 + z^2\right)^2}$$

$$\frac{\partial U}{\partial z} = \frac{2xz}{\left(x^2 + y^2 + z^2\right)^2}$$

Pentru a afla unghiul dintre doi gradienți, formăm produsul ...

$$\mathrm{grad}\,U_{(A)} \cdot \mathrm{grad}\,U_{(B)} = |\mathrm{grad}\,U_A||\mathrm{grad}\,U_B|\cos\alpha$$

$$\frac{\partial U_A}{\partial x} = \frac{7}{81}; \quad \frac{\partial U_A}{\partial y} = \frac{4}{81}; \quad \frac{\partial U_A}{\partial z} = \frac{4}{81}; \quad |\mathrm{grad}\,U_A| ...$$

$$\frac{\partial U_B}{\partial x} = -\frac{2}{25}; \quad \frac{\partial U_B}{\partial y} = -\frac{3}{50}; \quad \frac{\partial U_B}{\partial z} = 0; \quad |\mathrm{grad}\,U_B| ...$$

Din formula (2), folosind datele (3), avem:

$$\cos\alpha = -\frac{8}{9}.$$

6. Demonstrați formulele:

a) $\mathrm{grad}(U + c) = \mathrm{grad}\,U \quad (c - \mathrm{const})$;
b) $\mathrm{grad}\,cU = c\,\mathrm{grad}\,U$;
c) $\mathrm{grad}(U + V) = \mathrm{grad}\,U + \mathrm{grad}\,V$;
d) $\mathrm{grad}\,UV = V\mathrm{grad}\,U + U\mathrm{grad}\,V$;
e) $\mathrm{grad}\,U^2 = 2U\mathrm{grad}\,U$;
f) $\mathrm{grad}\,f(U) = f'(U)\mathrm{grad}\,U$.

Exemplu de rezolvare:

f)
$$\text{grad} f(U) = \frac{\partial f(U)}{\partial x}\vec{i} + \frac{\partial f(U)}{\partial y}\vec{j} + \frac{\partial f(U)}{\partial z}\vec{k} =$$
$$= \frac{\partial f}{\partial U}\frac{\partial U}{\partial x}\vec{i} + \frac{\partial f}{\partial U}\frac{\partial U}{\partial y}\vec{j} + \frac{\partial f}{\partial U}\frac{\partial U}{\partial z}\vec{k} = f'(U)\,\text{grad}\,U.$$

Celelalte - analogic.

7. Calculați:

a) $\text{grad}\, r$;

b) $\text{grad}\, r^2$;

c) $\text{grad}\,\frac{1}{r}$,

unde $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Rezolvare:

a)
$$\text{grad}\, r = \text{grad}\sqrt{x^2+y^2+z^2} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2}\,\vec{i} +$$
$$+\frac{\partial}{\partial y}\sqrt{x^2+y^2+z^2}\,\vec{j} + \frac{\partial}{\partial z}\sqrt{x^2+y^2+z^2}\,\vec{k} =$$
$$= \frac{1}{2\sqrt{x^2+y^2+z^2}}\left(2x\vec{i} + 2y\vec{j} + 2z\vec{k}\right) = \frac{\vec{r}}{r}.$$

Celelalte - analogic.

8. Calculați:

a) $\text{grad} f(r)$;

b) $\text{grad}(\vec{c}\cdot\vec{r})$.

Rezolvare:

a)
$$\text{grad}(\vec{c}\cdot\vec{r}) = \frac{\partial}{\partial x}(\vec{c}\cdot\vec{r})\vec{i} + \frac{\partial}{\partial y}(\vec{c}\cdot\vec{r})\vec{j} +$$
$$+\frac{\partial}{\partial z}(\vec{c}\cdot\vec{r})\vec{k} = \frac{\partial}{\partial x}(c_x\cdot x)\vec{i} + \frac{\partial}{\partial y}(c_y\cdot y)\vec{j} +$$
$$+\frac{\partial}{\partial z}(c_z\cdot z)\vec{k} = \vec{c}.$$

b) de sine stătător.

9. Calculați: $\text{grad}\left\{\left|\vec{c}\times\vec{r}\right|^2\right\}$.

$$\left[\vec{c}\times\vec{r}\right] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ c_x & c_y & c_z \\ x & y & z \end{vmatrix} = \vec{i}\left(c_y z - c_z y\right) +$$
$$+\vec{j}\left(c_z x - c_x z\right) + \vec{k}\left(c_x y - c_y x\right);$$
$$\left(\left[\vec{c}\times\vec{r}\right]\right)^2 = \left(c_y z - c_z y\right)^2 + \left(c_z x - c_x z\right)^2 + \left(c_x y - c_y x\right)^2;$$
$$\text{grad}\left\{\left|\vec{c}\times\vec{r}\right|^2\right\} = 2\left(c_z x - c_x z\right)c_z - 2\left(c_x y - c_y x\right)c_y + \ldots =$$
$$= 2\vec{r}(\vec{c}\cdot\vec{c}) - 2\vec{c}\cdot(\vec{c}\cdot\vec{r}).$$

10. Demonstrați formula:
$$\text{grad} f(U,V) = \frac{\partial f}{\partial U}\text{grad}\,U + \frac{\partial f}{\partial V}\text{grad}\,V;$$
$$\nabla^2(UV) = U\nabla^2 V + V\nabla^2 U + 2\nabla U\nabla V;$$

$$\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z};$$

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

11. Se să reprezinte sub formă de orţi câmpul: $\vec{a} = \vec{c} \times \mathrm{grad}\,U$, dacă $U = \mathrm{arctg}\,\dfrac{z}{\sqrt{x^2 + y^2}}$ şi $\vec{c} = \vec{i} + \vec{j} + \vec{k}$.

[illegible] staţionar

1. Demonstraţi

a) $\mathrm{div}(\vec{a} + \vec{b}) = \mathrm{div}\,\vec{a} + \mathrm{div}\,\vec{b}$

b) $\mathrm{div}(U\vec{c}) = \vec{c}\,\mathrm{grad}\,U$;

c) $\mathrm{div}(U\vec{a}) = U\mathrm{div}\,\vec{a} + \vec{a}\,\mathrm{grad}\,U$

Exemplu

b) $$\mathrm{div}(U\vec{c}) = \frac{\partial}{\partial x}(Uc_x) + \frac{\partial}{\partial y}(Uc_y) + \frac{\partial}{\partial z}(Uc_z) = \vec{c}\,\mathrm{grad}\,U.$$

2. Calculaţi: $\mathrm{div}(\mathrm{grad}\,U)$.

$$\mathrm{div}(\mathrm{grad}\,U) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}U\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}U\right) + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}U\right) = \Delta U.$$

3. Aflaţi $\mathrm{div}[\mathrm{grad} f(\vec{r})]$, unde $r = \sqrt{x^2 + y^2 + z^2}$.

În ce caz $\mathrm{div}[\mathrm{grad} f(\vec{r})] = 0$?

$$\mathrm{div}\left[\frac{\partial f(r)}{\partial r}\frac{\vec{r}}{r}\right] = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial r}\frac{x}{r}\right] + \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial r}\frac{y}{r}\right] + \frac{\partial}{\partial z}\left[\frac{\partial f}{\partial r}\frac{z}{r}\right] =$$

$$= \frac{\partial^2 f}{\partial r^2}\frac{\partial r}{\partial x}\frac{x}{r} + \frac{\partial f}{\partial r}\,\frac{r - \dfrac{x^2}{\sqrt{x^2 + y^2 + z^2}}}{r^2} + \ldots =$$

$$= \frac{\partial^2 f}{\partial r^2}\frac{x^2}{r^2} + \frac{\partial f}{\partial r}\frac{y^2 + z^2}{r^3} + \ldots =$$

$$= \frac{\partial^2 f}{\partial r^2} + \frac{\partial f}{\partial r}\frac{2}{r}.$$

Condiţia $\mathrm{div}[\mathrm{grad} f(\vec{r})] = 0$ ne conduce la o ecuaţie diferenţială