1. [4.135. Demidovici, p.436] Demonstrați identitățile:

a) $\mathrm{rot}(\vec{a} + \vec{b}) = \mathrm{rot}\,\vec{a} + \mathrm{rot}\,\vec{b}$;

b) $\mathrm{rot}(U\vec{a}) = U\mathrm{rot}\,\vec{a} + (\mathrm{grad}\,U \times \vec{a})$

Rezolvare:

b) Reprezentăm operatorul $\mathrm{rot}(U\vec{a})$ sub formă de determinant

$$\mathrm{rot}(U\vec{a}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ua_x & Ua_y & Ua_z \end{vmatrix} = \vec{i}\left(\frac{\partial}{\partial y}(Ua_z) - \frac{\partial}{\partial z}(Ua_y)\right) +$$

$$+\vec{j}\left(\frac{\partial}{\partial z}(Ua_x) - \frac{\partial}{\partial x}(Ua_z)\right) + \vec{k}\left(\frac{\partial}{\partial x}(Ua_y) - \frac{\partial}{\partial y}(Ua_x)\right) =$$

$$= \vec{i}\left(\frac{\partial U}{\partial y}a_z + U\frac{\partial a_z}{\partial y} - \frac{\partial U}{\partial z}a_y - \frac{\partial a_y}{\partial z}U\right) +$$

$$+\vec{j}\left(\frac{\partial U}{\partial z}a_x + U\frac{\partial a_x}{\partial z} - \frac{\partial U}{\partial x}a_z - \frac{\partial a_z}{\partial x}U\right) +$$

$$+\vec{k}\left(\frac{\partial U}{\partial x}a_y + U\frac{\partial a_y}{\partial x} - \frac{\partial U}{\partial y}a_x - \frac{\partial a_x}{\partial y}U\right) =$$

$$= \vec{i}\left(\frac{\partial U}{\partial y}a_z - \frac{\partial U}{\partial z}a_y\right) + \vec{i}\,U\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) +$$

$$+\vec{j}\left(\frac{\partial U}{\partial z}a_x - \frac{\partial U}{\partial x}a_z\right) + \vec{j}\,U\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) +$$

$$+\vec{k}\left(\frac{\partial U}{\partial x}a_y - \frac{\partial U}{\partial y}a_x\right) + \vec{k}\,U\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) =$$

$$= \mathbf{grad}\,U \times \vec{a} + U\,\mathbf{rot}\,\vec{a}.$$

2. Calculați a) $\mathrm{rot}\,\vec{r}$; b) $\mathrm{rot}(f(r)\vec{r})$.

Rezolvare:

b)

$$\mathrm{rot}(f(r)\vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(r)x & f(r)y & f(r)z \end{vmatrix} =$$

$$= \vec{i}\left(z\frac{\partial f(r)}{\partial r}\frac{y}{r} - y\frac{\partial f(r)}{\partial r}\frac{z}{r}\right) + \vec{j}\left(x\frac{\partial f(r)}{\partial r}\frac{z}{r} - z\frac{\partial f(r)}{\partial r}\frac{x}{r}\right) +$$

$$+\vec{k}\left(y\frac{\partial f(r)}{\partial r}\frac{x}{r} - x\frac{\partial f(r)}{\partial r}\frac{y}{r}\right) = \vec{0}.$$

3. Calculați mărimea și direcția $\mathrm{rot}\,\vec{a}$ în punctul $M(1, -2,$ [illegible]$)$, dacă:

$$\vec{a} = \frac{y}{z}\vec{i} + \frac{z}{x}\vec{j} + \frac{x}{y}\vec{k},$$

de sine stătător.

4. Calculați

a) $\mathrm{rot}\,\vec{c}f(r)$;

b) $\mathrm{rot}[\vec{c} \times f(r) \cdot \vec{r}]$,

$\vec{c}$ – vector constant.

Rezolvare:

a)

$$\mathrm{rot}(\vec{c} \cdot f(r)) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ c_x f(r) & c_y f(r) & c_z f(r) \end{vmatrix} =$$

$$= \vec{i}\left(\frac{\partial}{\partial y} c_z f(r) - \frac{\partial}{\partial z} c_y f(r)\right) + \vec{j}\left(\frac{\partial}{\partial z} c_x f(r) - \frac{\partial}{\partial x} c_z f(r)\right) +$$

$$+\vec{k}\left(\frac{\partial}{\partial x} c_y f(r) - \frac{\partial}{\partial y} c_x f(r)\right) = \vec{i}\left(c_z \frac{\partial f}{\partial r}\frac{y}{r} - c_y \frac{\partial f}{\partial r}\frac{z}{r}\right) +$$

$$+\vec{j}\left(c_x \frac{\partial f}{\partial r}\frac{z}{r} - c_z \frac{\partial f}{\partial r}\frac{x}{r}\right) + \vec{k}\left(c_y \frac{\partial f}{\partial r}\frac{x}{r} - c_x \frac{\partial f}{\partial r}\frac{y}{r}\right) =$$

$$= \frac{\partial f(r)}{\partial r}\frac{1}{r}\left[\vec{i}\left(c_z y - c_y z\right) + \vec{j}\left(c_z x - c_x z\right) + \vec{k}\left(c_x y - c_x y\right)\right] =$$

$$= \frac{\left[\vec{r} \times \vec{c}\right]}{r}\frac{\partial f(r)}{\partial r}$$

b) de sine stătător.

5. Demonstraţi identitatea:

$$\operatorname{div}\left(\vec{a} \times \vec{b}\right) = \vec{b}\operatorname{rot}\vec{a} - \vec{a}\operatorname{rot}\vec{b}.$$

Rezolvare:

$$\vec{a} \times \vec{b} = \vec{i}\left(a_y b_z - a_z b_y\right) + \vec{j}\left(a_z b_x - a_x b_z\right) + \vec{k}\left(a_x b_y - a_y b_x\right);$$

$$\frac{\partial}{\partial x}\left(a_y b_z - a_z b_y\right) = b_z \frac{\partial a_y}{\partial x} + a_y \frac{\partial b_z}{\partial x} - a_z \frac{\partial b_y}{\partial x} - b_y \frac{\partial a_z}{\partial x} =$$

$$= \left(b_z \frac{\partial a_y}{\partial x} - b_y \frac{\partial a_z}{\partial x}\right) + \left(a_y \frac{\partial b_z}{\partial x} - a_z \frac{\partial b_y}{\partial x}\right).$$

Primul termen reprezintă componenta x a produsului scalar $\vec{b}\operatorname{rot}\vec{a}$, al doilea – componenta x a mărimii $\vec{a}\operatorname{rot}\vec{b}$, la fel procedăm pentru derivatele parţiale după y şi z.

6. Să se afle:

a) $\operatorname{rot}\left(\operatorname{grad} U\right)$;

b) $\operatorname{div}\left(\operatorname{rot}\vec{a}\right)$,

de sine stătător.

VIII. CALCULUL SIMBOLIC AL MĂRIMILOR DIFERENŢIALE OPERATORI DIFERENŢIALI ∇ ŞI Δ

1. Calculaţi:

$$\operatorname{div}(\vec{c} \times \vec{r}) = \left(\nabla[\vec{c} \times \vec{r}]\right) = \nabla\left[\overset{\downarrow}{\vec{c}} \times \vec{r}\right] + \nabla\left[\vec{c} \times \overset{\downarrow}{\vec{r}}\right] =$$

$$= 0 - \nabla\left[\overset{\downarrow}{\vec{r}} \times \vec{c}\right] = -\vec{c}\left[\nabla \times \overset{\downarrow}{\vec{r}}\right] = -\vec{c}\operatorname{rot}\vec{r} = 0.$$

2.

$$\operatorname{div}(\vec{c} \times \vec{r}) = \left(\nabla[\vec{c} \times \vec{r}]\right) = \nabla\left[\overset{\downarrow}{\vec{c}} \times \vec{r}\right] + \nabla\left[\vec{c} \times \overset{\downarrow}{\vec{r}}\right] =$$

$$= 0 - \nabla\left[\overset{\downarrow}{\vec{r}} \times \vec{c}\right] = -\vec{c}\left[\nabla \times \overset{\downarrow}{\vec{r}}\right] = -\vec{c}\operatorname{rot}\vec{r} = 0;$$

$$\operatorname{div} r^2\vec{c} = \nabla\left(r^2 \cdot \vec{c}\right) = \nabla\left(\overset{\downarrow}{r^2} \cdot \vec{c}\right) + \nabla\left(r^2 \cdot \overset{\downarrow}{\vec{c}}\right) =$$

$$= \nabla\left(\overset{\downarrow}{r^2} \cdot \vec{c}\right) + 0 = \operatorname{grad} r^2 c = 2r\operatorname{grad} r\vec{c} =$$

$$= 2r\frac{\vec{r}}{r}\vec{c} = 2\left(\vec{r} \cdot \vec{c}\right);$$

3. $\operatorname{div}\varphi\vec{A} - ?$

$$\vec{A} = \vec{A}(x, y, z);$$

$$\varphi = \varphi(x, y, z);$$

$$\operatorname{div}\varphi\vec{A} = \nabla\left(\varphi\vec{A}\right) = \nabla\left(\overset{\downarrow}{\varphi}\vec{A}\right) + \nabla\left(\varphi\overset{\downarrow}{\vec{A}}\right) = \nabla \cdot \varphi\vec{A} + \left(\nabla \cdot \vec{A}\right) \cdot \varphi$$

$$= \vec{A}\operatorname{grad}\varphi + \varphi\operatorname{div}\vec{A}$$