

$$\begin{aligned}
 &= \bar{i} \left(\frac{\partial}{\partial y} c_y f(r) - \frac{\partial}{\partial z} c_z f(r) \right) + \bar{j} \left(\frac{\partial}{\partial z} c_z f(r) - \frac{\partial}{\partial x} c_x f(r) \right) + \\
 &+ \bar{k} \left(\frac{\partial}{\partial x} c_x f(r) - \frac{\partial}{\partial y} c_y f(r) \right) = \bar{i} \left(c_z \frac{\partial f}{\partial r} \frac{y}{r} - c_y \frac{\partial f}{\partial r} \frac{z}{r} \right) + \\
 &+ \bar{j} \left(c_x \frac{\partial f}{\partial r} \frac{z}{r} - c_z \frac{\partial f}{\partial r} \frac{x}{r} \right) + \bar{k} \left(c_y \frac{\partial f}{\partial r} \frac{x}{r} - c_x \frac{\partial f}{\partial r} \frac{y}{r} \right) = \\
 &= \frac{\partial f(r)}{\partial r} \frac{1}{r} \left[\bar{i}(c_z y - c_y z) + \bar{j}(c_x z - c_z x) + \bar{k}(c_x y - c_y x) \right] = \\
 &= \frac{[\bar{r} \times \bar{c}]}{r} \bar{c} f'(r)
 \end{aligned}$$

b) de sine stătător.

5. Demonstrați identitatea:

$$\operatorname{div}(\bar{a} \times \bar{b}) = \bar{b} \operatorname{rot} \bar{a} - \bar{a} \operatorname{rot} \bar{b}.$$

Rezolvare:

$$\begin{aligned}
 \bar{a} \times \bar{b} &= \bar{i}(a_y b_z - a_z b_y) + \bar{j}(a_z b_x - a_x b_z) + \bar{k}(a_x b_y - a_y b_x); \\
 \frac{\partial}{\partial x} (a_y b_z - a_z b_y) &= b_z \frac{\partial a_y}{\partial x} - a_z \frac{\partial b_y}{\partial x} + a_y \frac{\partial b_z}{\partial x} - a_x \frac{\partial b_z}{\partial x} - b_y \frac{\partial a_z}{\partial x} - \\
 &= \left(b_z \frac{\partial a_y}{\partial x} - b_y \frac{\partial a_z}{\partial x} \right) + \left(a_y \frac{\partial b_z}{\partial x} - a_z \frac{\partial b_y}{\partial x} \right).
 \end{aligned}$$

Primul termen reprezintă componenta x a produsului scalar $\bar{b} \operatorname{rot} \bar{a}$, al doilea - componenta x a mărimii $\bar{a} \operatorname{rot} \bar{b}$, la fel procedăm pentru derivatele parțiale după y și z .

6. Să se afle:

a) $\operatorname{rot}(\operatorname{grad} U)$;

b) $\operatorname{div}(\operatorname{rot} \bar{a})$,

de sine stătător.

1. Calculați:

$$\begin{aligned}
 \operatorname{div}(\bar{c} \times \bar{r}) &= (\nabla[\bar{c} \times \bar{r}]) = \nabla[\bar{c} \times \bar{r}] + \nabla[\bar{c} \times \bar{r}] = \\
 &= 0 - \nabla[\bar{r} \times \bar{c}] = -\bar{c}[\nabla \times \bar{r}] = -\bar{c} \operatorname{rot} \bar{r} = 0.
 \end{aligned}$$

2.

$$\begin{aligned}
 \operatorname{div}(\bar{c} \times \bar{r}) &= (\nabla[\bar{c} \times \bar{r}]) = \nabla[\bar{c} \times \bar{r}] + \nabla[\bar{c} \times \bar{r}] = \\
 &= 0 - \nabla[\bar{r} \times \bar{c}] = -\bar{c}[\nabla \times \bar{r}] = -\bar{c} \operatorname{rot} \bar{r} = 0;
 \end{aligned}$$

$$\begin{aligned}
 \operatorname{div} r^2 \bar{c} &= \nabla(r^2 \cdot \bar{c}) = \nabla(r^2 \cdot \bar{c}) + \nabla(r^2 \cdot \bar{c}) = \\
 &= \nabla(r^2 \cdot \bar{c}) + 0 = \operatorname{grad} r^2 \cdot \bar{c} = 2r \operatorname{grad} r \cdot \bar{c} = \\
 &= 2r \frac{\bar{r}}{r} \cdot \bar{c} = 2(\bar{r} \cdot \bar{c}).
 \end{aligned}$$

3. $\operatorname{div} \varphi \bar{A} = ?$

$$\bar{A} = \bar{A}(x, y, z);$$

$$\varphi = \varphi(x, y, z);$$

$$\begin{aligned}
 \operatorname{div} \varphi \bar{A} &= \nabla(\varphi \bar{A}) = \nabla(\varphi \bar{A}) + \nabla(\varphi \cdot \bar{A}) = \nabla \cdot \varphi \bar{A} + (\nabla \cdot \bar{A}) \cdot \varphi \\
 &= \bar{A} \operatorname{grad} \varphi + \varphi \operatorname{div} \bar{A}
 \end{aligned}$$

8.

$$\begin{aligned} \operatorname{div}((\vec{r} \times \vec{a}) \times \vec{c}) &= \nabla \cdot \left[\left[\begin{array}{c} \vec{r} \times \vec{a} \\ \downarrow \\ \vec{r} \end{array} \right] \times \vec{c} \right] = \nabla \cdot (\vec{r} \times \vec{c}) = \\ &= \vec{c}(\nabla \times \vec{r}) = \vec{c}[\nabla \times (\vec{r} \times \vec{a})] = \vec{c} \operatorname{rot}(\vec{r} \times \vec{a}) = \\ &= \vec{c}(-2\vec{a}) = -2\vec{a} \cdot \vec{c}. \end{aligned}$$

9.

$$\begin{aligned} \operatorname{div}((\vec{r} \times \vec{a}) \times \vec{r}) &= \nabla \cdot \left[\left[\begin{array}{c} \vec{r} \times \vec{a} \\ \downarrow \\ \vec{r} \end{array} \right] \times \vec{r} \right] = \nabla \cdot (\vec{r} \times \vec{r}) = \\ &= \nabla \cdot (\vec{r} \times \vec{r}) + \nabla \cdot (\vec{r} \times \vec{r}) = \vec{r}(\nabla \times \vec{r}) - \vec{r}(\nabla \times \vec{r}) = \\ &= \vec{r} \cdot \operatorname{rot}(\vec{r} \times \vec{a}) - \vec{r} \operatorname{rot} \vec{r} = \\ &= \vec{r} \operatorname{rot}(\vec{r} \times \vec{a}) = -2\vec{r} \cdot \vec{a}. \end{aligned}$$

10.

$$\begin{aligned} \operatorname{rot} \vec{r} \cdot (\vec{c} \cdot \vec{r}) &= [\nabla \times \vec{r}(\vec{c} \cdot \vec{r})] = \left[\nabla \times \vec{r}(\vec{c} \cdot \vec{r}) \right] + \\ &+ \left[\nabla \times \vec{r} \left(\begin{array}{c} \downarrow \\ \vec{c} \cdot \vec{r} \end{array} \right) \right] = (\vec{c} \cdot \vec{r}) \operatorname{rot} \vec{r} + \left[\nabla \left(\begin{array}{c} \downarrow \\ \vec{c} \cdot \vec{r} \end{array} \right) \right] \times \vec{r} = \\ &= 0 + [\operatorname{grad}(\vec{c} \cdot \vec{r}) \times \vec{r}] = [\vec{c} \times \vec{r}]. \end{aligned}$$

11.

$$\operatorname{rot}(\vec{c}(\vec{a} \cdot \vec{r})) = [\nabla \times \vec{c}(\vec{a} \cdot \vec{r})] =$$

4

$$\begin{aligned} \operatorname{div}(\vec{A} \times \vec{B}) &= \nabla \cdot (\vec{A} \times \vec{B}) = \nabla \cdot \left(\begin{array}{c} \downarrow \\ \vec{A} \times \vec{B} \end{array} \right) + \nabla \cdot (\vec{A} \times \vec{B}) = \\ &= B \left(\nabla \times \vec{A} \right) - \vec{A} \cdot \left(\nabla \times \vec{B} \right) = \vec{B} \operatorname{rot} \vec{A} - \vec{A} \operatorname{rot} \vec{B}. \end{aligned}$$

5.

$$\begin{aligned} \operatorname{rot}(\varphi \vec{A}) &= (\nabla \times \varphi \vec{A}) = \left(\nabla \times \varphi \vec{A} \right) + \left(\nabla \times \varphi \vec{A} \right) = \left(\nabla \varphi \times \vec{A} \right) + \\ &+ \left(\nabla \times \vec{A} \right) \varphi = [\operatorname{grad} \varphi \times \vec{A}] + \varphi \operatorname{rot} \vec{A}. \end{aligned}$$

6.

$$\begin{aligned} \operatorname{div} \vec{a}(\vec{c} \cdot \vec{r}) &= \nabla \vec{a}(\vec{c} \cdot \vec{r}) \cdot \vec{a}, \quad \vec{a}, \vec{c} = \text{const.} \\ \nabla \vec{a} \left(\begin{array}{c} \downarrow \\ \vec{c} \cdot \vec{r} \end{array} \right) &= \nabla \left(\begin{array}{c} \downarrow \\ \vec{a} \varphi \end{array} \right) = (\vec{a} \nabla \varphi) = (\vec{a} \nabla(\vec{c} \cdot \vec{r})) = \\ &= \vec{a} \operatorname{grad}(\vec{c} \cdot \vec{r}) = \vec{a} \cdot \vec{c}. \end{aligned}$$

7.

$$\begin{aligned} \operatorname{div}(\vec{r} \cdot (\vec{c} \cdot \vec{r})) &= \nabla(\vec{r} \cdot (\vec{c} \cdot \vec{r})) = \nabla \left(\begin{array}{c} \downarrow \\ \vec{r} \cdot (\vec{c} \cdot \vec{r}) \end{array} \right) + \\ &+ \nabla \left(\vec{r} \cdot \left(\begin{array}{c} \downarrow \\ \vec{c} \cdot \vec{r} \end{array} \right) \right) = (\nabla \cdot \vec{r})(\vec{c} \cdot \vec{r}) + \vec{r} \nabla(\vec{c} \cdot \vec{r}) = \\ &= \operatorname{div} \vec{r}(\vec{c} \cdot \vec{r}) + \vec{r} \operatorname{grad}(\vec{c} \cdot \vec{r}) = 3(\vec{c} \cdot \vec{r}) + (\vec{c} \cdot \vec{r}) = 4(\vec{c} \cdot \vec{r}). \end{aligned}$$

IX. FORMULA OSTROGRADSKI-GAUSS

12.

$$\begin{aligned}
 &= [\text{grad}(\vec{a} \cdot \vec{r}) \times \vec{c}] = [\vec{a} \times \vec{c}] \\
 \text{rot}[(\vec{c} \times \vec{r}) \times \vec{a}] &= \nabla \times \left[\left[\vec{c} \times \vec{r} \right] \times \vec{a} \right] = \\
 &= \left[\nabla \times \left[\vec{r} \times \vec{a} \right] \right] - \left[\nabla \times \left[\vec{a} \times \vec{r} \right] \right] = \\
 &= - \left[\nabla \times \left[\vec{a} \times \left[\vec{c} \times \vec{r} \right] \right] \right] = \left[\nabla \times \left[\vec{a} \times \left[\vec{r} \times \vec{c} \right] \right] \right] = \\
 &= \left[\nabla \times (\vec{r}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{r})) \right] = \left[\nabla \times \vec{r}(\vec{a} \cdot \vec{c}) \right] - \\
 &= \left[\nabla \times \vec{c}(\vec{a} \cdot \vec{r}) \right] = 0 - \left[\nabla(\vec{a} \cdot \vec{r}) \times \vec{c} \right] = -[\vec{a} \times \vec{c}] = \vec{c} \times \vec{a}.
 \end{aligned}$$

13.

$$\text{rot}[(\vec{c} \times \vec{r}) \times \vec{r}] = 3\vec{c} \times \vec{r}.$$

1. Să se calculeze fluxul vectorului \vec{r} :
 a) prin suprafața laterală a conului $x^2 + y^2 \leq z^2$ ($0 \leq z \leq h$);

b) prin baza acestui con.

Rezolvare:

a) De sine statător.

b).

$$x^2 + y^2 \leq z^2 \quad (0 \leq z \leq h);$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

$$\Phi = \iint_S a_n dS.$$

După definiția fluxului:

Folosind teorema Ostrogradski-Gauss vom avea:

$$\begin{aligned}
 \iint_S a_n dS &= \iiint_V \text{div} \vec{a} dx dy dz = \iiint_V \text{div} \vec{r} dx dy dz = \\
 &= \iiint_V 3 dx dy dz = 3 \int_0^h \int_0^{\sqrt{z^2-y^2}} \int_0^{\sqrt{z^2-y^2-x^2}} dz dy dx = \\
 &= 3 \int_0^h \int_0^{y_r} \int_0^{x_r} dx dy dz = 3 \int_0^h dz \int_0^{y_r} \int_0^{x_r} \sqrt{z^2 - y^2} dy dx = \\
 &= 3 \int_0^h dz \int_0^{y_r} \sqrt{z^2 - y^2} dy = 3 \int_0^h dz \int_0^{z \cos \alpha} z \sin \alpha dy = \\
 &= 3 \int_0^h dz \int_0^{z \cos \alpha} z \sin \alpha dz = 3 \int_0^h dz \int_0^{z \cos \alpha} z^2 dz = 3 \int_0^h dz \left[\frac{z^3}{3} \right]_0^{z \cos \alpha} = \\
 &= \int_0^h z^3 \cos^3 \alpha dz = \int_0^h z^3 \cos \alpha (1 - \sin^2 \alpha) dz = \int_0^h z^3 \cos \alpha dz - \int_0^h z^3 \cos \alpha \sin^2 \alpha dz = \\
 &= \int_0^h z^3 \cos \alpha dz - \int_0^h z^3 \cos \alpha \sin^2 \alpha dz = \int_0^h z^3 \cos \alpha dz - \int_0^{\pi/2} \int_0^h z^3 \cos \alpha dz \sin^2 \alpha d\alpha = \\
 &= \int_0^h z^3 \cos \alpha dz \left[\alpha \right]_0^{\pi/2} - \int_0^{\pi/2} \int_0^h z^3 \cos \alpha dz \sin^2 \alpha d\alpha = \int_0^h z^3 \cos \alpha dz \left[\frac{\pi}{2} - 0 \right] - \int_0^{\pi/2} \left[\frac{z^4}{4} \right]_0^h \sin^2 \alpha d\alpha = \\
 &= \frac{\pi}{2} \int_0^h z^3 dz - \frac{h^4}{4} \int_0^{\pi/2} \sin^2 \alpha d\alpha = \frac{\pi}{2} \cdot \frac{h^4}{4} - \frac{h^4}{4} \int_0^{\pi/2} \sin^2 \alpha d\alpha = \frac{\pi h^4}{8} - \frac{h^4}{4} \int_0^{\pi/2} \sin^2 \alpha d\alpha =
 \end{aligned}$$