$$= \vec{i}\left(\frac{\partial}{\partial y} c_z f(r) - \frac{\partial}{\partial z} c_y f(r)\right) + \vec{j}\left(\frac{\partial}{\partial z} c_x f(r) - \frac{\partial}{\partial x} c_z f(r)\right) +$$

$$+\vec{k}\left(\frac{\partial}{\partial x} c_y f(r) - \frac{\partial}{\partial y} c_x f(r)\right) = \vec{i}\left(c_z \frac{\partial f}{\partial r}\frac{y}{r} - c_y \frac{\partial f}{\partial r}\frac{z}{r}\right) +$$

$$+\vec{j}\left(c_x \frac{\partial f}{\partial r}\frac{z}{r} - c_z \frac{\partial f}{\partial r}\frac{x}{r}\right) + \vec{k}\left(c_y \frac{\partial f}{\partial r}\frac{x}{r} - c_x \frac{\partial f}{\partial r}\frac{y}{r}\right) =$$

$$= \frac{\partial f(r)}{\partial r}\frac{1}{r}\left[\vec{i}(c_z y - c_y z) + \vec{j}(c_z x - c_x z) + \vec{k}(c_x y - c_x y)\right] =$$

$$= \frac{[\vec{r} \times \vec{c}]}{r}\frac{\partial f(r)}{\partial r}$$

b) de sine stătător.

5. Demonstrați identitatea:

$$\operatorname{div}(\vec{a} \times \vec{b}) = \vec{b}\operatorname{rot}\vec{a} - \vec{a}\operatorname{rot}\vec{b}.$$

Rezolvare:

$$\vec{a} \times \vec{b} = \vec{i}(a_y b_z - a_z b_y) + \vec{j}(a_z b_x - a_x b_z) + \vec{k}(a_x b_y - a_y b_x);$$

$$\frac{\partial}{\partial x}(a_y b_z - a_z b_y) = b_z \frac{\partial a_y}{\partial x} + a_y \frac{\partial b_z}{\partial x} - a_z \frac{\partial b_y}{\partial x} - b_y \frac{\partial a_z}{\partial x} =$$

$$= \left(b_z \frac{\partial a_y}{\partial x} - b_y \frac{\partial a_z}{\partial x}\right) + \left(a_y \frac{\partial b_z}{\partial x} - a_z \frac{\partial b_y}{\partial x}\right).$$

Primul termen reprezintă componenta x a produsului scalar $\vec{b}\operatorname{rot}\vec{a}$, al doilea – componenta x a mărimii $\vec{a}\operatorname{rot}\vec{b}$, la fel procedăm pentru derivatele parțiale după y și z.

6. Să se afle:

a) $\operatorname{rot}(\operatorname{grad} U)$;

b) $\operatorname{div}(\operatorname{rot}\vec{a})$,

de sine stătător.

VIII. CALCULUL SIMBOLIC AL MĂRIMILOR DIFERENȚIALE OPERATORI DIFERENȚIALI ∇ ȘI Δ

1. Calculați:

$$\operatorname{div}(\vec{c} \times \vec{r}) = (\nabla[\vec{c} \times \vec{r}]) = \nabla\left[\overset{\downarrow}{\vec{c}} \times \vec{r}\right] + \nabla\left[\vec{c} \times \overset{\downarrow}{\vec{r}}\right] =$$

$$= 0 - \nabla\left[\overset{\downarrow}{\vec{r}} \times \vec{c}\right] = -\vec{c}\left[\nabla \times \overset{\downarrow}{\vec{r}}\right] = -\vec{c}\operatorname{rot}\vec{r} = 0.$$

2.

$$\operatorname{div}(\vec{c} \times \vec{r}) = (\nabla[\vec{c} \times \vec{r}]) = \nabla\left[\overset{\downarrow}{\vec{c}} \times \vec{r}\right] + \nabla\left[\vec{c} \times \overset{\downarrow}{\vec{r}}\right] =$$

$$= 0 - \nabla\left[\overset{\downarrow}{\vec{r}} \times \vec{c}\right] = -\vec{c}\left[\nabla \times \overset{\downarrow}{\vec{r}}\right] = -\vec{c}\operatorname{rot}\vec{r} = 0;$$

$$\operatorname{div} r^2\vec{c} = \nabla(r^2 \cdot \vec{c}) = \nabla\left(\overset{\downarrow}{r^2} \cdot \vec{c}\right) + \nabla\left(r^2 \cdot \overset{\downarrow}{\vec{c}}\right) =$$

$$= \nabla\left(\overset{\downarrow}{r^2} \cdot \vec{c}\right) + 0 = \operatorname{grad} r^2 c = 2r\operatorname{grad} r\vec{c} =$$

$$= 2r\frac{\vec{r}}{r}\vec{c} = 2(\vec{r} \cdot \vec{c}).$$

3.

$$\operatorname{div}\varphi\vec{A} - ?$$

$$\vec{A} = \vec{A}(x, y, z);$$

$$\varphi = \varphi(x, y, z);$$

$$\operatorname{div}\varphi\vec{A} = \nabla(\varphi\vec{A}) = \nabla\left(\overset{\downarrow}{\varphi}\vec{A}\right) + \nabla\left(\varphi\overset{\downarrow}{\vec{A}}\right) = \nabla \cdot \varphi\vec{A} + (\nabla \cdot \vec{A}) \cdot \varphi$$

$$= \vec{A}\operatorname{grad}\varphi + \varphi\operatorname{div}\vec{A}$$

4.

$$\operatorname{div}(\vec A\times\vec B)=\nabla(\vec A\times\vec B)=\nabla\left(\overset{\downarrow}{\vec A}\times\vec B\right)+\nabla\left(\vec A\times\overset{\downarrow}{\vec B}\right)=$$

$$=\vec B\left(\nabla\times\overset{\downarrow}{\vec A}\right)-\vec A\left(\nabla\times\overset{\downarrow}{\vec B}\right)=\vec B\operatorname{rot}\vec A-\vec A\operatorname{rot}\vec B.$$

5.

$$\operatorname{rot}\varphi\vec A=(\nabla\times\varphi\vec A)=\left(\nabla\times\overset{\downarrow}{\varphi}\vec A\right)+\left(\nabla\times\varphi\overset{\downarrow}{\vec A}\right)=\left(\nabla\overset{\downarrow}{\varphi}\times\vec A\right)+$$

$$+\left(\nabla\times\overset{\downarrow}{\vec A}\right)\varphi=\left[\operatorname{grad}\varphi\times\vec A\right]+\varphi\operatorname{rot}\vec A.$$

6.

$$\operatorname{div}\vec a(\vec c\cdot\vec r)=\nabla\vec a(\vec c\cdot\vec r)\quad \vec a,\ \vec c\ \text{ const.}$$

$$\nabla\vec a\Big(\underbrace{\vec c\cdot\overset{\downarrow}{\vec r}}_{\varphi}\Big)=\nabla\left(\vec a\,\overset{\downarrow}{\varphi}\right)=(\vec a\nabla\varphi)=(\vec a\nabla(\vec c\cdot\vec r))=$$

$$=\vec a\operatorname{grad}(\vec c\cdot\vec r)=\vec a\cdot\vec c.$$

7.

$$\operatorname{div}(\vec r\cdot(\vec c\cdot\vec r))=\nabla(\vec r\cdot(\vec c\cdot\vec r))=\nabla\left(\overset{\downarrow}{\vec r}\cdot(\vec c\cdot\vec r)\right)+$$

$$+\nabla\left(\vec r\cdot\left(\vec c\cdot\overset{\downarrow}{\vec r}\right)\right)=(\nabla\cdot\vec r)(\vec c\cdot\vec r)+\vec r\nabla(\vec c\cdot\vec r)=$$

$$=\operatorname{div}\vec r(\vec c\cdot\vec r)+\vec r\operatorname{grad}(\vec c\cdot\vec r)=3(\vec c\cdot\vec r)+(\vec c\cdot\vec r)=4(\vec c\cdot\vec r).$$

8.

$$\operatorname{div}((\vec r\times\vec a)\times\vec c)=\nabla\Big[\Big[\underbrace{\overset{\downarrow}{\vec r}\times\vec a}_{\vec f}\Big]\times\vec c\Big]=\nabla\left(\overset{\downarrow}{\vec f}\times\vec c\right)=$$

$$=\vec c(\nabla\times\vec f)=\vec c[\nabla\times[\vec r\times\vec a]]=\vec c\operatorname{rot}(\vec r\times\vec a)=$$

$$=\vec c(-2\vec a)=-2\vec a\cdot\vec c.$$

9.

$$\operatorname{div}((\vec r\times\vec a)\times\vec r)=\nabla\Big[\Big[\underbrace{\vec r\times\vec a}_{\vec f}\Big]\times\vec r\Big]=\nabla(\vec f\times\vec r)=$$

$$=\nabla\left(\overset{\downarrow}{\vec f}\times\vec r\right)+\nabla\left(\vec f\times\overset{\downarrow}{\vec r}\right)=\vec r\left(\nabla\times\overset{\downarrow}{\vec f}\right)-\vec f\left[\nabla\times\overset{\downarrow}{\vec r}\right]=$$

$$=\vec r\cdot\operatorname{rot}[\vec r\times\vec a]-\vec f\operatorname{rot}\vec r=$$

$$=\vec r\operatorname{rot}(\vec r\times\vec a)=-2\vec r\cdot\vec a.$$

10.

$$\operatorname{rot}(\vec r\cdot(\vec c\cdot\vec r))=[\nabla\times\vec r(\vec c\cdot\vec r)]=\left[\nabla\times\overset{\downarrow}{\vec r}(\vec c\cdot\vec r)\right]+$$

$$+\left[\nabla\times\vec r\left(\vec c\cdot\overset{\downarrow}{\vec r}\right)\right]=(\vec c\cdot\vec r)\operatorname{rot}\vec r+\left[\nabla\left(\vec c\cdot\overset{\downarrow}{\vec r}\right)\times\vec r\right]=$$

$$=0+\left[\operatorname{grad}(\vec c\cdot\vec r)\times\vec r\right]=[\vec c\times\vec r].$$

11.

$$\operatorname{rot}(\vec c(\vec a\cdot\vec r))=[\nabla\times\vec c(\vec a\cdot\vec r)]=$$

$$= \left[grad(\vec{a}\cdot\vec{r})\times\vec{c}\right] = [\vec{a}\times\vec{c}].$$

12.

$$\mathrm{rot}[(\vec{c}\times\vec{r})\times\vec{a}] = \left[\nabla\times\left[\left[\underbrace{\vec{c}\times\overset{\downarrow}{\vec{r}}}_{\vec{f}}\right]\times\vec{a}\right]\right] =$$

$$= \left[\nabla\times\left[\overset{\downarrow}{\vec{f}}\times\vec{a}\right]\right] - \left[\nabla\times\left[\vec{a}\times\overset{\downarrow}{\vec{f}}\right]\right] =$$

$$= -\left[\nabla\times\left[\vec{a}\times\left[\vec{c}\times\overset{\downarrow}{\vec{r}}\right]\right]\right] = \left[\nabla\times\left[\vec{a}\times\left[\overset{\downarrow}{r}\times\vec{c}\right]\right]\right] =$$

$$= \left[\nabla\times\left(\vec{r}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{r})\right)\right] = \left[\nabla\times\overset{\downarrow}{\vec{r}}(\vec{a}\cdot\vec{c})\right] -$$

$$-\left[\nabla\times\vec{c}\left(\vec{a}\cdot\overset{\downarrow}{\vec{r}}\right)\right] = 0 - \left[\nabla\left(\vec{a}\cdot\overset{\downarrow}{\vec{r}}\right)\times\vec{c}\right] = -[\vec{a}\times\vec{c}] = \vec{c}\times\vec{a}.$$

13.

$$\mathrm{rot}\left[(\vec{c}\times\vec{r})\times\vec{r}\right] = 3\vec{c}\times\vec{r}.$$

IX. FORMULA OSTROGRADSKI-GAUSS

1. Să se calculeze fluxul vectorului $\vec{r}$:
a) prin suprafaţa laterala a conului

$$x^2 + y^2 \le z^2 \quad (0 \le z \le h);$$

b) prin baza acestui con.

Rezolvare:

a) De sine stătător.

b).

$$x^2 + y^2 \le z^2 \quad (0 \le z \le h);$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

După definiţia fluxului:

$$\Phi = \iint_S a_n dS.$$

Folosind teorema Ostrogradski-Gauss vom avea :

$$\iint_S a_n dS = \iiint_V \mathrm{div}\vec{a}\,dxdydz = \iiint_V \mathrm{div}\vec{r}\,dxdydz =$$

$$= \iiint_V 3dxdydz = 3\int_0^z \int_0^{\sqrt{z^2-y^2}} \int_0^x dzdydx =$$

$$= 3\int_0^v dz\int_0^y dy\int_0^{\sqrt{z^2-y^2}} dx = 3\int_0^h dz\int_0^y \sqrt{z^2-y^2}\,dy =$$

$$= 3\int_0^h dz\int_0^y \sqrt{z^2 - z^2\cos^2\alpha}\,z\sin\alpha\,dy =$$

$$= 3\int_0^h dz\int_0^{z\cos\alpha} -z\sin\alpha\, z\sin\alpha\, d\alpha =$$

$$= 3\int_0^h z^2dz\int_0^{z\cos\alpha}\sin^2\alpha\,d\alpha = -3\int_0^h z^2dz\left(-\int_0^{\pi/2}\sin^2\alpha\,d\alpha\right) =$$