$$= \left[grad(\vec a \cdot \vec r) \times \vec c\right] = \left[\vec a \times \vec c\right].$$

12.

$$\mathrm{rot}\left[(\vec c \times \vec r) \times \vec a\right] = \left[\nabla \times \left[\left[\underbrace{\vec c \times \overset{\downarrow}{\vec r}}_{\vec f}\right] \times \vec a\right]\right] =$$

$$= \left[\nabla \times \left[\overset{\downarrow}{\vec f} \times \vec a\right]\right] - \left[\nabla \times \left[\vec a \times \overset{\downarrow}{\vec f}\right]\right] =$$

$$= -\left[\nabla \times \left[\vec a \times \left[\vec c \times \overset{\downarrow}{\vec r}\right]\right]\right] = \left[\nabla \times \left[\vec a \times \left[\overset{\downarrow}{\vec r} \times \vec c\right]\right]\right] =$$

$$= \left[\nabla \times \left(\vec r(\vec a \cdot \vec c) - \vec c(\vec a \cdot \vec r)\right)\right] = \left[\nabla \times \overset{\downarrow}{\vec r}(\vec a \cdot \vec c)\right] -$$

$$-\left[\nabla \times \vec c\left(\vec a \cdot \overset{\downarrow}{\vec r}\right)\right] = 0 - \left[\nabla\left(\vec a \cdot \overset{\downarrow}{\vec r}\right) \times \vec c\right] = -\left[\vec a \times \vec c\right] = \vec c \times \vec a.$$

13.

$$\mathrm{rot}\left[(\vec c \times \vec r) \times \vec r\right] = 3\vec c \times \vec r.$$

IX. FORMULA OSTROGRADSKI-GAUSS

1. Să se calculeze fluxul vectorului $\vec r$:
a) prin suprafaţa laterala a conului

$$x^2 + y^2 \le z^2 \quad (0 \le z \le h);$$

b) prin baza acestui con.
Rezolvare:
a) De sine stătător.
b).

$$x^2 + y^2 \le z^2 \quad (0 \le z \le h);$$

$$\vec r = x\vec i + y\vec j + z\vec k.$$

După definiţia fluxului:

$$\Phi = \iint_S a_n dS.$$

Folosind teorema Ostrogradski-Gauss vom avea :

$$\iint_S a_n dS = \iiint_V \mathrm{div}\vec a\, dxdydz = \iiint_V \mathrm{div}\vec r\, dxdydz =$$

$$= \iiint_V 3dxdydz = 3\int_0^z \int_0^{\sqrt{z^2-y^2}} \int_0^x dzdydx =$$

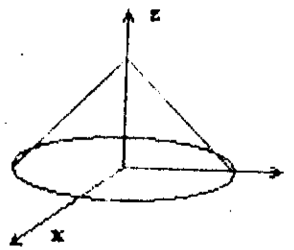
$$= 3\int_0^z dz \int_0^y dy \int_0^{\sqrt{z^2-y^2}} dx = 3\int_0^h dz \int_0^y \sqrt{z^2 - y^2}\, dy =$$

$$= 3\int_0^h dz \int_0^y \sqrt{z^2 - z^2\cos^2\alpha}\, z\sin\alpha\, dy =$$

$$= 3\int_0^h dz \int_0^{z\cos\alpha} -z\sin\alpha\, z\sin\alpha\, d\alpha =$$

$$= 3\int_0^h z^2 dz \int_0^{z\cos\alpha} \sin^2\alpha\, d\alpha = -3\int_0^h z^2 dz \left(-\int_0^{\pi/2} \sin^2\alpha\, d\alpha\right) =$$

$$= 3\int_0^h z^2 dz \int_0^{\pi/2} \left(\frac{1-\cos 2\alpha}{2}\right) d\alpha = \pi h^3;$$



sau, dacă am fi știut volumul conului, am fi scris:

$$\Phi = \iiint_V div\vec{r}\, dV_{con} = 3\iiint dV_{con} = 3V_{con}.$$

2. Să se calculeze fluxul vectorului $\vec{a} = \vec{i}yz + \vec{j}xz + \vec{k}xy$,

a) prin suprafața laterală a cilindrului $x^2 + y^2 \le a^2$, iar $(0 \le z \le h)$;

b) prin suprafața totală a lui.

Rezolvare (de sine stătător).

3. Aflați fluxul razei vectoare $\vec{r}$, prin suprafața descrisă de ecuația:

$$z = 1 - \sqrt{x^2 + y^2} \quad (0 \le z \le 1).$$

Rezolvare (de sine stătător).

4. Să se calculeze fluxul vectorului

$$\vec{a} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$$

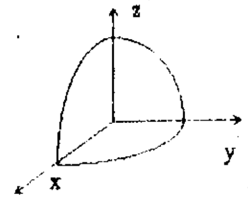
prin octantul pozitiv al sferei

$$x^2 + y^2 + z^2 = 1, x \le 0, y \le 0, z \le 0.$$

Rezolvare:

$$\iint \vec{a} \cdot \vec{n} dS = \iiint_V \mathrm{div}\vec{a}\, dV;$$

$$\mathrm{div}\vec{a} = 2(x + y + z);$$



$$\Phi = \int_0^1 dz \int_0^z dx \int_0^{\sqrt{z^2-y^2}} dy\big(2(x + y + z)\big) =$$

$$= 2\int_0^1 dz \int_0^z dx \int_0^{\sqrt{z^2-y^2}} (x + y + z) dy =$$

$$= 2\int_0^1 dz \int_0^z dx \left[\int_0^{\sqrt{z^2-x^2}} x dy + \int_0^{\sqrt{z^2-x^2}} y dy + \int_0^{\sqrt{z^2-x^2}} z dy\right] =$$

$$= 2\int_0^1 dz \int_0^z dx \left[x\sqrt{z^2 - x^2} + z\sqrt{z^2 - x^2} + \frac{z^2 - x^2}{2}\right] = 2\int_0^1 dz \times$$

$$\times \left[\int_0^z x\sqrt{z^2 - x^2} dx + \int_0^z z\sqrt{z^2 - x^2} dx + \int_0^z \frac{z^2 - x^2}{2} dx\right].$$

$$I_1 = \int_0^z x\sqrt{z^2 - x^2} dx = -\frac{1}{2}\int_{z^2}^0 \sqrt{z^2 - x^2}\, d(z^2 - x^2) =$$

$$= -\frac{1}{2}\int_0^{z^2} \sqrt{z^2 - x^2}\, d(z^2 - x^2) =$$

$$= \frac{1}{2}\int_0^{z^2}\sqrt{t}\,dt = \frac{1}{2}\frac{t^{3/2}}{3/2}\Bigg|_0^{z^2} = \frac{\left(z^2\right)^{3/2}}{3} = \frac{z^3}{3};$$

$$I_2 = \int_0^z z\sqrt{z^2 - x^2}\,dx = \int_0^z z\sqrt{z^2 - z^2\cos^2\alpha}\,d(z\cos\alpha) =$$

$$= z\int_0^z z\sqrt{1-\cos^2\alpha}\,z(-\sin\alpha)d\alpha = -z^3\int_0^z\sqrt{1-\cos^2\alpha}\,(-\sin\alpha)d\alpha =$$

$$= -z^3\int_{\pi/2}^{0}\sin^2\alpha\,d\alpha = z^3\int_0^{\pi/2}\frac{1-\cos 2\alpha}{2}\,d\alpha =$$

$$= z^3\left[\int_0^{\pi/2}\frac{1}{2}d\alpha - \int_0^{\pi/2}\frac{\cos 2\alpha}{2}\frac{d(2\alpha)}{2}\right] = \frac{\pi}{4}z^3;$$

$$I_3 = \frac{z^3}{6} - \frac{z^3}{2} = -\frac{z^3}{3}.$$

5. Aflați fluxul vectorului

$$\vec{A} = (x-1)^3\vec{i} + (y+2)^3\vec{j} + (z-2)^3\vec{k}$$

prin suprafața: $(x-1)^2 + (y+2)^2 + (z-2)^2 = R^2$.

Rezolvare:

$$\operatorname{div}A = 3(x-1)^2 + 3(y+2)^2 + 3(z-2)^2;$$

$$\Phi = \iint_S \vec{A}d\vec{S} = \iiint_V \operatorname{div}\vec{A}\,dV =$$

$$= 3\iiint_V\left((x-1)^2 + (y+2)^2 + (z-2)^2\right)dV.$$

În acest caz e mai comod de trecut la coordonate sferice:

$$\Phi = 3\iiint_V r^2 r^2\sin\theta\,d\theta\,d\varphi\,dr = 3\int_0^R r^4 dr\int_0^\pi \sin\theta\,d\theta\int_0^{2\pi}d\varphi = \frac{12\pi r^5}{5}.$$

6. Să se calculeze fluxul vectorului :

$$\vec{a} = y\vec{i} + z\vec{j} + x\vec{k}$$

prin suprafața totală a piramidei, mărginită de suprafețele: $x=0$, $y=0$, $z=0$, $x+y+z=a$ $(a>0)$.

Rezolvare (de sine stătător).

7. Aflându-se în mișcare, lichidul necomprimabil umple volumul V. Considerând că în regiunea lui lipsesc surse și scurgeri, deduceți ecuația continuității.

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\vec{v}) = 0,$$

unde $\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\vec{v}) = 0$ - densitatea lichidului, $\vec{v}$ - vectorul vitezei, t - timpul.